Dirichlet series

Definition: Let $f \in \mathcal{A}$ and $s \in \mathbb{C}$. We define the Dirichlet series attached to f to be the formal series

$$L_f(s) := \sum_{n \in \mathbb{N}} \frac{f(n)}{n^s}.$$

Convention: Oftentimes in analytic number theory, if $s \in \mathbb{C}$, we write $s = \sigma + it$, where $\sigma = \mathrm{Re}(s)$ and $t = \mathrm{Im}(s)$.

Half planes of (absolute) convergence

Theorem: Let $f \in \mathcal{A}$. There exists $\sigma_a \in \mathbb{R} \cup \{\pm\infty\}$ such that $L_f(s)$ converges absolutely for for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \sigma_a(f)$ and it does not converge absolutely for $\mathrm{Re}(s) < \sigma_a(f)$.

$\sigma_a(f)$ is called abscissa of absolute convergence.

Proof: Let $\Delta = \{ s \in \mathbb{C} : L_f(s) \text{ converges absolutely} \}$.
If $\Delta = \emptyset$, then set $\sigma_a = \infty$, nothing to prove.

We define $\sigma_a(f) := \inf \{ \mathrm{Re}(s) : s \in \Delta \} \in \mathbb{R} \cup \{-\infty\}$.

We need to show that if $\sigma = \mathrm{Re}(s) > \sigma_a(f)$,
then $L_f(s)$ converges absolutely.

Let $\sigma > \sigma_a(f)$. Then there exists $\sigma_a(f) \leq \sigma' < \sigma$
and $s' = \sigma' + it'$ such that
$L_f(s')$ absolutely convergent. (s' exists by definition)

Then $\sum_{n \in \mathbb{N}} \left| \frac{f(n)}{n^s} \right| = \sum_{n \in \mathbb{N}} \frac{|f(n)|}{n^\sigma} < \sum_{n \in \mathbb{N}} \frac{|f(n)|}{n^{\sigma'}} < \infty$.

$\Rightarrow L_f(s)$ absolutely convergent. ☒.

Theorem: Let $f \in \mathcal{A}$. There exists $\sigma_c(f) \in \mathbb{R} \cup \{\pm\infty\}$ (called abscissa of conditional convergence) such that $L_f(s)$ converges if $\sigma > \sigma_c(f)$ and $L_f(s)$ does not converge if $\sigma < \sigma_c(f)$.
The convergence is uniform on compact sets
and $\sigma_a(f) - 1 \leq \sigma_c(f) \leq \sigma_a(f)$.

$\mathrm{Re}(s) > \sigma_a(f)$
$\Rightarrow L_f(s)$ abs conv

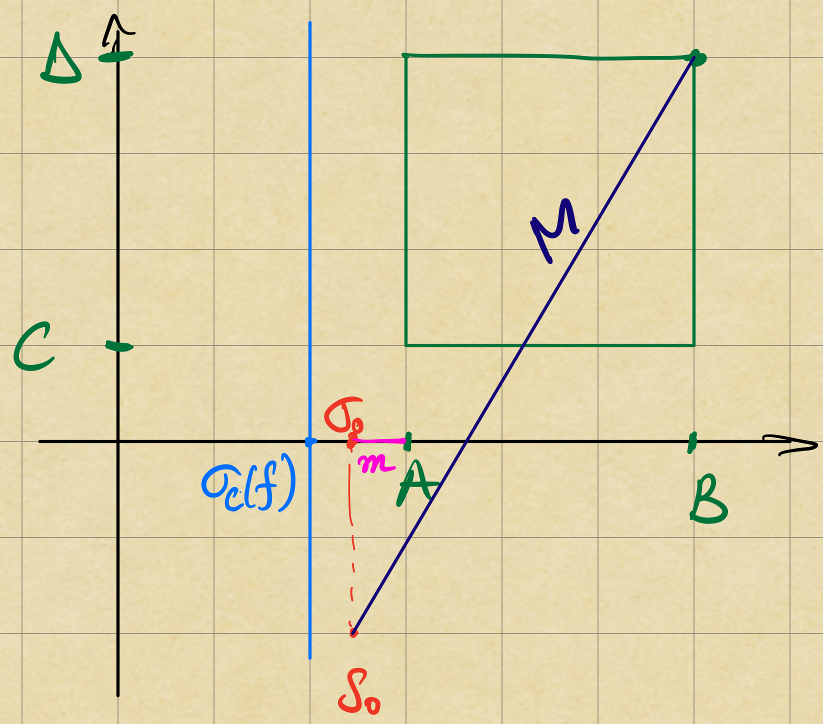
$\mathrm{Re}(s) > \sigma_c(f)$
$\Rightarrow L_f(s)$ conv

Proof: If $\{s \in \mathbb{C} : L_f(s) \text{ convergent}\} = \phi$,
Set $\sigma_c(f) = \infty$ and nothing to prove.

Otherwise, set $\sigma_c(f) = \inf\{\mathrm{Re}(s) : L_f(s) \text{ converges}\}$.

Let $K = [A, B] \times [C, D]$ a compact rectangle inside $\{s \in \mathbb{C} : \mathrm{Re}(s) > \sigma_c(f)\}$. Note that every compact set inside $\{\mathrm{Re}(s) > \sigma_c(f)\}$ is contained in such rectangle. There fore it is enough to prove uniform convergence on K.

By definition, $\exists\, s_0 \in \left\{ \sigma_c(f) \le \mathrm{Re}(s) < A : L_f(s) \text{ converges} \right\}$



Let $s \in K$. Define $S(s,y,x) := \sum_{y<n\leq x} \frac{f(n)}{n^s}$

$$S_0(y,x) := \sum_{y<n\leq x} \frac{f(n)}{n^{s_0}}.$$

Fix $\varepsilon > 0$. As $L_f(s_0)$ converges, $\exists\, Y_0 > 1$ such that $\forall\ x \geq y \geq y_0$, we have $|S_0(y,x)| \leq \varepsilon$. (by Cauchy criterion)

By partial summation,

$$S(s,y,x) = \sum_{y<n\leq x} \frac{f(n)}{n^{s_0}} \cdot n^{s_0-s} =$$

$$= S_0(y,x)\cdot x^{s_0-s} - (s_0-s)\int_y^x S_0(y,\xi)\,\xi^{s_0-s-1}\,d\xi$$

Set $m := \min\{\sigma - \sigma_0 : s \in K\} = A - \sigma_0 > 0.$

$M := \max\{|s - s_0| : s \in K\} > 0.$

$$|x^{s_0 - s}| = x^{\sigma_0 - \sigma} \leq x^{-m}$$

$$\int_y^x |\xi^{s_0 - s - 1}| \, d\xi = \int_y^x \xi^{\sigma_0 - \sigma - 1} d\xi = \left[\frac{\xi^{\sigma_0 - \sigma}}{\sigma_0 - \sigma}\right]_y^x \leq \frac{y^{-m}}{m}.$$

Hence for all $x > y \geq y_0$ and $s \in K$,

$$|S(s, y, x)| \leq \varepsilon \cdot x^{-m} + \varepsilon \cdot \frac{M}{m} y^{-m} \leq \varepsilon\left(1 + \frac{M}{m}\right).$$

This shows $\sum_{n=1}^{N} \frac{f(n)}{n^s}$ is Cauchy and uniformly convergent inside K.

It remains to show $\sigma_a(f) - 1 \leq \sigma_c(f) \leq \sigma_a(f)$.

RHS is clear.

Let $s \in \mathbb{C}$ such that $\sigma = \mathrm{Re}(s) > \sigma_c(f)$. We want to show $\sigma + 1 > \sigma_a(f)$.

Let $s' \in \mathbb{C}$ s.t. $\sigma > \sigma' > \sigma_c(f)$ and $L_f(s')$ conv.

Set $\delta := \sigma - \sigma'$.

$L_f(s')$ convergent $\Rightarrow \exists\, N_0 \in \mathbb{N}$ s.t. $\forall n \geq n_0$, $\left|\frac{f(n)}{n^{s'}}\right| \leq 1$.

$$\Rightarrow \sum_{n \geq N_0} \left|\frac{f(n)}{n^{s+L}}\right| = \sum_{n \geq N_0} \frac{|f(n)|}{n^{\sigma+L}} = \sum_{n \geq N_0} \frac{|f(n)|}{n^{\sigma'}} \cdot \frac{1}{n^{1+\sigma}}$$

$$\leq \sum_{n \geq N_0} \frac{1}{n^{2+\sigma}} < \infty.$$

$\Rightarrow L_f(s+L)$ absolutely convergent. $\boxtimes$

<u>Holomorphicity of Dirichlet series in half-plane of conditional convergence</u>

Recall the following theorem from complex analysis:

<u>Thm (Weierstrass)</u>:

Let $\Omega \subseteq \mathbb{C}$ open, $f_n : \Omega \to \mathbb{C}$ sequence of holomorphic functions on Ω. Assume that pointwise limit $f = \lim_{n\to\infty} f_n$ exists and that convergence is uniform on compacta.

Then f is holomorphic and moreover $f_n' \xrightarrow{n\to\infty} f'$ uniformly on compact sets.

Corollary: Let $f \in \mathcal{A}$. Then L_f is holomorphic on $\{s \in \mathbb{C} : Re(s) > \sigma_c(f)\}$ and

$$L_f'(s) = \sum_{n=1}^{\infty} - \frac{f(n) \log n}{n^s} \text{ on } Re(s) > \sigma_c(f).$$

Proof: Let $f_N(s) := \sum_{n=1}^{N} \frac{f(n)}{n^s}$.

This is holomorphic on $\{s \in \mathbb{C} : Re(s) > \sigma_c(f)\}$ and $f_N \to f$ uniformly on compacta inside $\Omega = \{s \in \mathbb{C} : Re(s) > \sigma_c(f)\}$.

Also clearly $f_N'(s) = -\sum_{n=1}^{N} \frac{f(n) \log n}{n^s}$. $\square$

Algebraic properties of L-functions

Thm: Suppose $f, g \in \mathcal{A}$ and L_f, L_g both converge absolutely at s. Then L_{f*g} converges absolutely at s and $L_{f*g}(s) = L_f(s) L_g(s)$.

Proof: $$\sum_{n \geq 1} \left| \frac{f*g(n)}{n^s} \right| \leq \sum_{n \in \mathbb{N}} \sum_{n_1 n_2 = n} \frac{|f(n_1) g(n_2)|}{n^\sigma}$$

$$\leq \sum_{n\in\mathbb{N}} \sum_{n_1 n_2 = n} \left|\frac{f(n_1)}{n_1^s}\right| \left|\frac{g(n_2)}{n_2^s}\right|$$

$$= \left(\sum_{n_1\in\mathbb{N}} \left|\frac{f(n_1)}{n_1^s}\right|\right)\left(\sum_{n_2\in\mathbb{N}} \left|\frac{g(n_2)}{n_2^s}\right|\right) < \infty.$$

hence $L_{f*g}(s)$ absolutely convergent at s if $L_f(s)$ & $L_g(s)$ are abs conv at s.

Moreover, note that

$$L_{f*g}(s) = \sum_{n\in\mathbb{N}} \sum_{n_1 n_2 = n} \frac{f(n_1)}{n_1^s} \frac{g(n_2)}{n_2^s} = L_f(s) L_g(s)$$

in the region of absolute convergence. ☒.

Corollary: $\sigma_a(f*g) \leq \max(\sigma_a(f), \sigma_a(g))$.

Thm: (Identity theorem).

Suppose $f, g \in \mathcal{A}$ and $\max\{\sigma_a(f), \sigma_a(g)\} < \sigma_0 < \infty$.

Suppose that $L_f(s) = L_g(s)$ for $Re(s) \geq \sigma_0$.

Then $f = g$.

Proof: Let $h = f - g \in \mathcal{A}$.

Then L_h conv abs for $Re(s) > \sigma_0$ and $L_h(s) = 0$.

Let $n_0 = \inf\{n : h(n) \neq 0\}$.

If $n_0 \neq \infty$, we have $\dfrac{h(n_0)}{n_0^s} = -\sum_{n > n_0} \dfrac{h(n)}{n^s}$

(for $Re(s) \geq \sigma_0$).

Thus,
$$|h(n_0)| \leq \sum_{n>n_0} |h(n)| \left(\frac{n_0}{n}\right)^{\sigma}$$
$$= \sum_{n>n_0} |h(n)| \left(\frac{n_0}{n}\right)^{\sigma_0} \left(\frac{n_0}{n}\right)^{\sigma - \sigma_0}$$
$$\leq \sum_{n>n_0} |h(n)| \left(\frac{n_0}{n}\right)^{\sigma} \left(\frac{n_0}{n_0+2}\right)^{\sigma - \sigma_0}$$
$$= \underbrace{n_0^{\sigma_0}}_{\text{indep of } s} \underbrace{\left(\frac{n_0}{n_0+2}\right)^{\sigma-\sigma_0}}_{\to 0 \text{ as } \sigma \to \infty} \underbrace{\sum_{n>n_0} \frac{|h(n)|}{n^{\sigma_0}}}_{\text{indep of } s}$$

$\Longrightarrow h(n_0) = 0$, which is absurd. $\blacksquare$

Informally, $L_f(s)$ determines uniquely $f(n)$ for $Re(s) > \sigma_a(f)$ and moreover we know $L_{f*g}(s) = L_f(s) L_g(s)$ uniquely identifies $f*g$ on $Re(s) > \max\{\sigma_a(f), \sigma_a(g)\}$.

Examples:

- $\zeta(s) = \sum_{n \in \mathbb{N}} \frac{1}{n^s} = L_{\mathcal{E}}(s)$

Exercise: $\sigma_a(\mathcal{E}) = \sigma_c(\mathcal{E}) = 1$

- $L_\tau(s) = L_{\mathcal{E}*\mathcal{E}}(s) = \zeta^2(s)$ conv abs for $\mathrm{Re}(s) > 1$

- $L_e(s) = 1$

- $L_\mu(s) = L_e(s) L_{\mathcal{E}}(s)^{-1} = \frac{1}{\zeta(s)}$, for $\mathrm{Re}(s) > 1$.

Since L_μ holomorphic on $\mathrm{Re}(s) > 1$, it follows that $\zeta(s) \neq 0$ for $\mathrm{Re}(s) > 1$.

- $L_{\log}(s) = \sum_{n \in \mathbb{N}} \frac{\log n}{n^s} = -L_{\mathcal{E}}'(s) = -\zeta'(s)$, for $\mathrm{Re}(s) > 1$.

- $L_\Lambda(s) = L_{\log * \mu}(s) = -\frac{\zeta'(s)}{\zeta(s)}$, $\mathrm{Re}(s) > 1$.

- $L_{id}(s) = \sum_{n \in \mathbb{N}} \frac{1}{n^{s-1}} = \zeta(s-1)$, for $\mathrm{Re}(s) > 2$.

- $L_\varphi(s) = L_\mu(s) L_{id}(s) = \frac{\zeta(s-1)}{\zeta(s)}$, for $\mathrm{Re}(s) > 2$.